A disentanglement relation for SU(3) coherent states

1989 J. Phys. A: Math. Gen. 22 L1089

(http://iopscience.iop.org/0305-4470/22/23/002)

J. Phys. A: Math. Gen. **22** (1989) L1089–L1092. Printed in the UK

LETTER TO THE EDITOR

A disentanglement relation for SU(3) coherent states

K Raghunathan, M Seetharaman and S S Vasan

Department of Theoretical Physics, University of Madras, Guindy Campus, Madras—600 025, India

Received 20 September 1989

Abstract. A Baker–Campbell–Hausdorff-type disentanglement relation relevant to SU(3) coherent states is presented.

We present a disentanglement relation, also known as the Baker–Campbell–Hausdorff formula, for the SU(3) group. Similar results already available in the literature for the SU(2) group (Gilmore 1974), Arrechi *et al* (1972) have been found (Chaturvedi *et al* 1987) to be very useful in the context of coherent states, calculation of Berry's phase, etc. We believe that our results for SU(3) will also be useful in such contexts.

Our aim is to establish a disentanglement relation having the form

$$e^M = \exp(\bar{\alpha}E_{31})\exp(\bar{\beta}E_{32})\exp(\bar{\gamma}E_{21})\exp(h_1 \ln \delta)\exp(h_2 \ln \varepsilon)$$
$$\times \exp(\gamma E_{12})\exp(\beta E_{23})\exp(\alpha E_{13}) \qquad (1)$$

with

$$M = z_1^* E_{31} + z_2^* E_{32} + z_3^* E_{21} - z_1 E_{13} - z_2 E_{23} - z_3 E_{12}. \qquad (2)$$

In (2) the E_{ij} $(i \neq j)$ are the set of three creation and three annihilation operators for SU(3) and h_1 and h_2 are diagonal operators given by $h_1 = E_{11} - E_{22}$ and $h_2 = E_{11} + E_{22} + 2E_{33}$. The E_{ij} satisfy the commutation relations

$$[E_{ij}, E_{kl}] = E_{il}\delta_{jk} - E_{jk}\delta_{il}. \qquad (3)$$

The z_i in (1) are complex variables in terms of which we must find α, $\bar{\alpha}$, β, $\bar{\beta}$, γ, $\bar{\gamma}$, δ and ε. The relation (1) is solely a consequence of the Lie algebra of the group and therefore the solution for α, $\bar{\alpha}\ldots$ is representation independent. Hence we can use any faithful representation to obtain them. We make use of a fundamental 3×3 matrix representation of SU(3) in which E_{ij} are given by 3×3 matrices, namely,

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}. \qquad (4)$$

Let us consider the LHS of (1) in this representation. It is easy to see that M satisfies the minimal equation

$$M^3 = -R^2 M + c\mathbb{1} \qquad (5a)$$

where

$$R^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 \qquad \text{and} \qquad c = z_1^* z_2 z_3 - z_1 z_2^* z_3^*. \qquad (5b)$$

In view of this, we can write for any positive integer k,

$$M^k = U_k M^2 + V_k M + W_k \mathbb{1} \qquad (6)$$

where U_k, V_k and W_k are polynomials in R^2 and c. Explicit forms of U, V and W can be obtained by noting that they satisfy the coupled recurrence relations

$$U_{k+1} = V_k \qquad V_{k+1} = -R^2 U_k + W_k \qquad W_{k+1} = cU_k \tag{7}$$

with $U_0 = 0$, $V_0 = 0$ and $W_0 = 1$. The result (7) follows from the fact that

$$\begin{aligned} M^{k+1} &= U_{k+1} M^2 + V_{k+1} M + W_{k+1} \mathbb{1} = U_k M^3 + V_k M^2 + W_k M \\ &= V_k M^2 + (-R^2 U_k + W_k) M + c U_k \mathbb{1}. \end{aligned} \tag{8}$$

Since $e^M = \Sigma_k M^k/k!$, we need to find $\Sigma\, U_k/k!$, $\Sigma\, V_k/k!$ and $\Sigma\, W_k/k!$. Let us define, for this purpose,

$$F(\lambda) = \sum_k \frac{\lambda^k U_k}{k!}. \tag{9}$$

It follows from (7) that the U_k satisfy the recurrence relation

$$U_{k+1} = -R^2 U_{k-1} + cU_{k-2} \tag{10}$$

with $U_0 = U_1 = 0$ and $U_2 = 1$. Hence, multiplying (10) by λ^{k+1} and summing over k, we find after some rearrangement that

$$\sum_{k=0}^{\infty} \lambda^k U_k = S(\lambda) = \frac{\lambda^2}{1 + \lambda^2 R^2 - \lambda^3 c}. \tag{11}$$

$F(\lambda)$ can be now evaluated from this result using the Laplace transform technique. Let $f(s)$ be the Laplace transform of $F(\lambda)$. We have

$$\begin{aligned} f(s) &= \mathscr{L}F(\lambda) = \sum_k U^k/s^{k+1} \\ &= (1/s)S(1/s) = \frac{1}{s^3 + sR^2 - c}. \end{aligned} \tag{12}$$

Now,

$$\begin{aligned} F(\lambda) &= \mathscr{L}^{-1} f(s) = \mathscr{L}^{-1} \frac{1}{s^3 + sR^2 - c} \\ &= (-1) \frac{(s_2 - s_3)\, e^{-s_1\lambda} + (s_3 - s_1)\, e^{-s_2\lambda} + (s_1 - s_2)\, e^{-s_3\lambda}}{(s_1 - s_2)(s_2 - s_3)(s_3 - s_1)} \end{aligned} \tag{13}$$

in which $-s_1$, $-s_2$ and $-s_3$ are the roots of the cubic $s^3 + sR^2 - c = 0$ and are given by

$$s_j = -i\alpha \cos\left(\vartheta + \frac{2\pi(j-1)}{3}\right)$$

where $\alpha = (4/3)^{1/2} R$ and $\cos 3\vartheta = (4ic/\alpha^3)$. From a knowledge of $F(\lambda)$ one can evaluate the quantities

$$G(\lambda) = \sum \lambda^k V_k/k! \qquad \text{and} \qquad H(\lambda) = \sum \lambda^k W_k/k!$$

needed to evaluate $e^{\lambda M}$. Since $G(\lambda) = dF/d\lambda$, we get at once

$$G(\lambda) = \frac{s_1(s_2 - s_3)\, e^{-s_1\lambda} + s_2(s_3 - s_1)\, e^{-s_2\lambda} + s_3(s_1 - s_2)\, e^{-s_3\lambda}}{(s_1 - s_2)(s_2 - s_3)(s_3 - s_1)}. \tag{14}$$

Similarly,

$$H(\lambda)=1+c\int_0^\lambda \mathrm{d}\lambda\, F(\lambda)$$

$$=(-1)\frac{s_2s_3(s_2-s_3)\,\mathrm{e}^{-s_1\lambda}+s_3s_1(s_3-s_1)\,\mathrm{e}^{-s_2\lambda}+s_1s_2(s_1-s_2)\,\mathrm{e}^{-s_3\lambda}}{(s_1-s_2)(s_2-s_3)(s_3-s_1)}. \tag{15}$$

In obtaining (15) we have used $c=-s_1s_2s_3$. We can now write the LHS of (1) (setting $\lambda=1$) as

$$\begin{aligned}\mathrm{e}^M=&[-(|z_1|^2+|z_3|^2)F+H]E_{11}-(z_1z_2^*F+z_3G)E_{12}+(z_3z_2F-z_1G)E_{13}\\&-(z_2z_1^*F-z_3^*G)E_{21}+[-(|z_2|^2+|z_3|^2)F+H]E_{22}\\&-(z_1z_3^*F+z_2G)E_{23}+(z_2^*z_3^*F+z_1^*G)E_{31}\\&-(z_1^*z_3F-z_2^*G)E_{32}+[-(|z_1|^2+|z_2|^2)F+H]E_{33}.\end{aligned} \tag{16}$$

Evaluating the RHS of (1) is simple because in the representation chosen the expotentials involving the diagonal operators h_1 and h_2 can be written as

$$\begin{aligned}&\exp(h_1\ln\delta)=\delta E_{11}+(1/\delta)E_{22}+E_{33}\\&\exp(h_2\ln\varepsilon)=\varepsilon E_{11}+\varepsilon E_{22}+(1/\varepsilon^2)E_{33}\end{aligned} \tag{17}$$

and for the rest of the exponentials we can use McLaurin's expansion and use the fact that $E_{ij}^2=0$ in the chosen representation so that

$$\exp(\eta E_{ij})=1+\eta E_{ij} \qquad (i\neq j). \tag{18}$$

Thus, the RHS of (1) can now be multiplied out and it becomes

$$\begin{aligned}\mathrm{e}^M=&\varepsilon\delta E_{11}+\gamma\varepsilon\delta E_{12}+\varepsilon\delta(\alpha+\beta\gamma)E_{13}+\bar{\gamma}\varepsilon\delta E_{21}\\&+[\gamma\bar{\gamma}\varepsilon\delta+(\varepsilon/\delta)]E_{22}+[\bar{\gamma}\varepsilon\delta(\alpha+\beta\gamma)+(\beta\varepsilon/\delta)]E_{23}\\&+\varepsilon\delta(\bar{\alpha}+\bar{\beta}\bar{\gamma})E_{31}+[\gamma\varepsilon\delta(\bar{\alpha}+\bar{\beta}\bar{\gamma})+(\bar{\beta}\varepsilon/\delta)]E_{32}\\&+[\varepsilon\delta(\alpha+\beta\gamma)(\bar{\alpha}+\bar{\beta}\bar{\gamma})+(\beta\bar{\beta}\varepsilon/\delta)+(1/\varepsilon^2)]E_{33}.\end{aligned} \tag{19}$$

On equating (16) and (19) we get the following set of equations:

$$\varepsilon\delta=H-(|z_1|^2+|z_3|^2)F \tag{20}$$

$$\gamma\varepsilon\delta=-(z_1z_2^*F+z_3G) \tag{21}$$

$$\varepsilon\delta(\alpha+\beta\gamma)=(z_3z_2F-z_1G) \tag{22}$$

$$\bar{\gamma}\varepsilon\delta=-(z_2z_1^*F-z_3^*G) \tag{23}$$

$$\gamma\bar{\gamma}\varepsilon\delta+(\varepsilon/\delta)=H-(|z_2|^2+|z_3|^2)F \tag{24}$$

$$\bar{\gamma}\varepsilon\delta(\alpha+\beta\gamma)+(\beta\varepsilon/\delta)=-(z_1z_3^*F+z_2G) \tag{25}$$

$$\varepsilon\delta(\bar{\alpha}+\bar{\beta}\bar{\gamma})=(z_3^*z_2^*F+z_1^*G) \tag{26}$$

$$\gamma\varepsilon\delta(\bar{\alpha}+\bar{\beta}\bar{\gamma})+(\bar{\beta}\varepsilon/\delta)=-(z_1^*z_3F-z_2^*G) \tag{27}$$

$$\varepsilon\delta(\alpha+\beta\gamma)(\bar{\alpha}+\bar{\beta}\bar{\gamma})+(\beta\bar{\beta}\varepsilon/\delta)+(1/\varepsilon^2)=H-(|z_2|^2+|z_2|^2)F. \tag{28}$$

Even though we have nine equations to solve for eight parameters, the ninth equation is not independent of the rest and can be shown to follow from the rest of the equations due to the fact that $\det(e^M)=1$. The explicit solutions of the algebraic equations (20)-(27) can be found to be the following:

$$\gamma=-(z_1z_2^*F+z_3G)/[H-(|z_1|^2+|z_3|^2)F] \tag{29}$$

$$\varepsilon^2=H^2+|z_3|^2(R^2F^2+G^2-HF)-R^2HF-cFG \tag{30}$$

$$\delta^2=[H-(|z_1|^2+|z_3|^2)F]^2/\varepsilon^2 \tag{31}$$

$$\beta=[z_1z_3^*(R^2F^2+G^2-HF)+z_2(cF^2-HG)]/\varepsilon^2 \tag{32}$$

$$\begin{aligned}\alpha=\{&z_2z_3[FH^2-HG^2-HF^2(2R^2-|z_2|^2)+FG^2(R^2-|z_3|^2)+R^2F^3(R^2-|z_2|^2)]\\&-z_1[GH^2+FGH(R^2+|z_2|^2)-cHF^2+2cFG^2\\&+cF^3(R^2-|z_2|^2)]\}/[H-(|z_1|^2+|z_3|^2)F]\varepsilon^2.\end{aligned} \tag{33}$$

$\bar\alpha$, $\bar\beta$ and $\bar\gamma$ are obtained by replacing z_i by $-z_i^*$ in the above. We can verify (28) using the above solutions and the fact that

$$\det e^M=c^2F^3+cG^3+H^3+cR^2F^2G+R^4F^2H+R^2G^2H-2R^2H^2F-3cFGH=1.$$

The applications of our results, mentioned already, such as Berry's phase for SU(3) coherent states, treatment of three level atoms, etc, are under investigation.

References

Gilmore R 1974 *Lie Groups, Lie Algebras, and Some of their Applications* (New York: Wiley)

Arrechi F T, Courtens E, Gilmore R and Thomas H 1972 *Phys. Rev.* A **6** 2211

Chaturvedi S, Sriram M S and V Srinivasan 1987 *J. Phys. A: Math. Gen.* **20** L1071